Teaching STEM Math to first year college students

Keynote address: http://huichawaii.org/wp-content/uploads/2017/06/2017-STEM-Book-June-02.pdf
Backup: http://jontalle.web.engr.illinois.edu/uploads/298/ProgramBook-STEM.17.pdf

Prof. Jont B Allen
Elec. Comp. Eng.
University of IL, Urbana, USA

http://jontalle.web.engr.illinois.edu/uploads/298/

June 8, 2017

Goals for: "A review of mathematics, via its history."

- First year college students take calculus in high school
- >30–40% advance-place (AP) out of Calc-I & II
  - Due to poor fundamentals, they struggle in engineering courses
  - Mathematics courses (Calc-III, Linear Alg., DiffEq, . . .) are a mystery
- Solution: *Concepts in Mathematical–Physics based & its History*
  - Proven to work, and students love it:
    - "I have to wonder why it isn't the standard way of teaching mathematics to engineers."
    - "Fourier series and Laplace transforms and distributions are now understandable, even easy."
    - "Engineering courses are now 'easy' after ECE298ja"
    - "Learning complex analysis makes math less 'magical.' "
    - "Homeworks are hard, but worth the effort"
    - "ECE298ja students are #1 in their Math and Engineering classes
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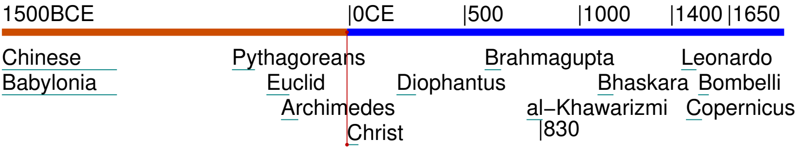
UIUC Student body

The number of undergraduate students in Electrical and Computer Engineering

- 420 ECE undergraduate students/year
- 50% electrical and 50% computer engineering
- Undergraduate enrollment: 420*4.5 $\approx$ 1900
- $\approx$100 ECE transfers
- Other engineering departments (Mech, Civil, MatSci) have somewhat smaller enrollment

Time-line: 5000 BCE–1650 CE



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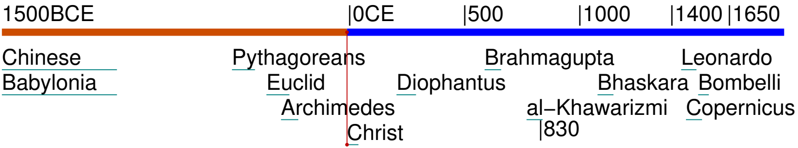

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Key concepts in math and physics

- Fundamental theorems
  - Integers may be factored into primes (FT Arith)
  - Density of primes within integers (PNT)
  - Algebra (factoring polynomials)
  - Integral Calculus (Real and complex integration)
  - Vector calculus (Helmholtz Theorem)
- Other key theorems:
  - Complex analytic functions
  - Calculus in the complex plane
  - Cauchy Integral Theorem (Residue integration)
  - Riemann sphere (defining the point at ∞)
- Applications:
  - Linear algebra
  - Difference, scalar & vector differential equations
  - Maxwell's vector differential equations

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The three Streams and their mathematics Stillwell [2010]

- *The Pythagorean Theorem bore three streams:*
- *2–3 Centuries per stream:*

1) **Numbers:**

6^{th}BCE $\mathbb{N}$ *counting numbers*, $\mathbb{Q}$ *(Rationals)*, $\mathbb{P}$ *Primes*
5^{th}BCE $\mathbb{Z}$ *Common Integers*, $\mathbb{I}$ *Irrationals*
7^{th}CE *zero* $\in \mathbb{Z}$

2) **Geometry:** *(e.g., lines, circles, spheres, toroids, ...)*

17^{th}CE *Composition of polynomials (Descartes, Fermat)*
Euclid's Geometry $+$ *algebra* $\Rightarrow$ *Analytic Geometry*
18^{th}CE *Fundamental Theorem of Algebra*

3) **Infinity:** *($\infty \rightarrow$ Sets)*

17-18^{th}CE *Taylor series, analytic functions, calculus (Newton)*
19^{th}CE $\mathbb{R}$ *Real*, $\mathbb{C}$ *Complex 1851; Open vs. closed Sets 1874*

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Fundamental theorems of:

1. **Number systems: Stream 1**
   - arithmetic (FTA)
   - prime number (PNT)
2. **Geometry: Stream 2**
   - algebra
   - Bézout
3. **Calculus: Stream 3**
   - Leibniz $\mathbb{R}^1$ (area under a curve only depends on end points)
   - complex $\mathbb{C} \subset \mathbb{R}^2$ (area under a curve only depends on end points!)
   - vectors $\mathbb{R}^3, \mathbb{R}^n, \mathbb{R}^\infty$
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Stream 1: WEEK 2-10, Lects 2-10

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- Stream 2: Algebraic Equations (WEEK 4-8, Lect 11-22)
- Stream 3a: Scalar Differential Equations (WEEK 8-12, Lect 23-34)
- Stream 3b: Partial Differential Equations (WEEK 12-14, Lect 35-42)

Famous problems in number theory (Stream 1)

- Finding prime numbers using sieves
- Continued fraction algorithm (rational approximations of irrational numbers)
- Pythagorean triplets (integer solutions of $c^2 = a^2 + b^2$)

Finding prime numbers: the sieve of Eratosthenes

1. Write N integers from 2 to $N-1$. Set $k=1$. The first element $\pi_1 = 2$ is prime. Cross out $n \cdot \pi_n$: (e.g., $n \cdot \pi_1 = 4, 8, 16, 32, \cdots$).

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There are 15 primes less than $N=50$: $\pi_k = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$.

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Continued fraction algorithm (CFA)

Given an irrational number $x \in \mathbb{I}$, $n/m = CFA(x)$ finds a rational approximation $n/m \in \mathbb{Q}$, to any desired accuracy.

Examples:

$$\widehat{\pi}_1 \approx 3 + \frac{1}{7 + 0.0625\ldots} \approx 3 + \frac{1}{7} = \frac{22}{7}$$

$$\widehat{\pi}_2 \approx 3 + 1/(7 + 1/16) = 3 + 16/113 = 355/113$$

$$\widehat{e_5} = 3 + 1/(-4 + 1/(2 + 1/(5 + 1/(-2 + 1/(-7))))) - 1.753610^{-6}$$

Pythagorean triplets & Euclid's formula

Find $a, b, c \in \mathbb{N}$ such that

$$c^2 = a^2 + b^2.$$

Solution: Set $p > q \in \mathbb{N}$. Then (Euclid's formula)

$$c = p^2 + q^2, \qquad a = p^2 - q^2, \qquad b = 2pq. \tag{1}$$

This result may be directly verified

$$[p^2 + q^2]^2 = [p^2 - q^2]^2 + [2pq]^2$$

or

$$p^4 + q^4 + \cancel{2p^2q^2} = p^4 + q^4 - \cancel{2p^2q^2} + \cancel{4p^2q^2}.$$

Deriving Euclid's formula (Eq. 1) is obviously more difficult.

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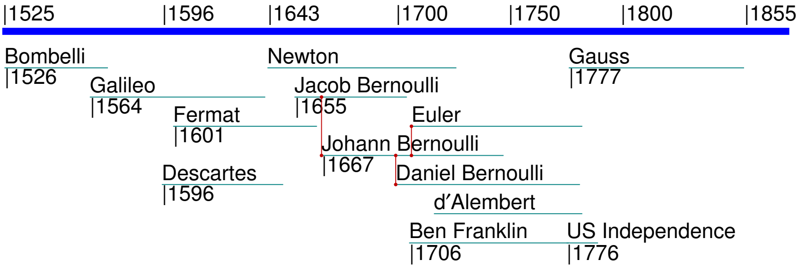
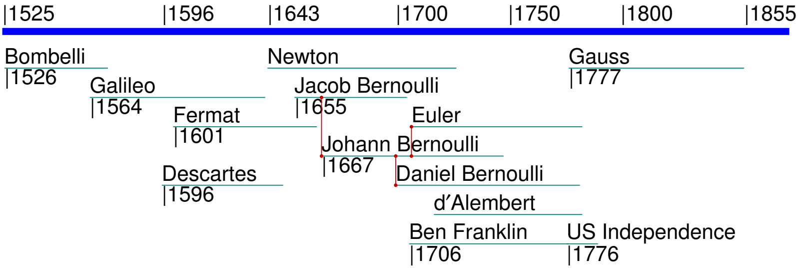
Time-line: Bombelli–Gauss 16-18 centuries



- Newton, Bernoulli family, Euler, d'Alembert and Gauss
- Johann teaches mathematics to Euler and Daniel
- Euler's technique dominates mathematics for 200 years (examples)
- d'Alembert proposes:
  - general solution to scalar wave equation
  - fundamental theorem of algebra (FTA)
- Gauss had great conceptual depth (PNT, Least squares, FFT)

Time-line: Bombelli–Gauss 16-18 centuries



- Newton, Bernoulli family, Euler, d'Alembert and Gauss
- Johann teaches mathematics to Euler and Daniel
- Euler's technique dominates mathematics for 200 years (examples)
- d'Alembert proposes:
  - general solution to scalar wave equation
  - fundamental theorem of algebra (FTA)
- Gauss had great conceptual depth (PNT, Least squares, FFT)

Analytic functions and Taylor series (Newton)

An *analytic function* is one that may be expanded in a power series

- Geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

- This is easily seen to be correct by cross-multiplying

$$1 = (1-x)\sum x^n = \sum_{n=0}^{\infty} x^n - \sum_{n=1}^{\infty} x^n = 1$$

- The *Taylor series* is much more powerful

$$f(x) = \sum_n \underbrace{\frac{1}{n!}\frac{d^n}{dx^n} f(x)}_{a_n}\bigg|_{x=0} x^n$$

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Jakob Bernoulli #1 (1654-1705)

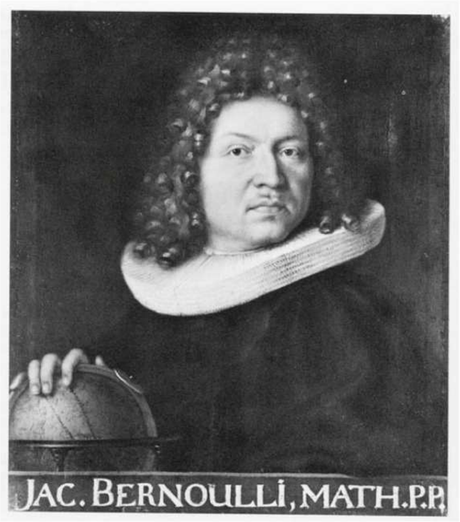

Figure 13.10: Portrait of Jakob Bernoulli by Nicholas Bernoulli

Johann Bernoulli (#2) 10^{th} child; Euler's advisor

Figure 13.11: Johann Bernoulli

Leonhard Euler, most prolific of all mathematicians

Figure 10.4: Leonhard Euler

Euler's sieve and the zeta function: $\zeta(s)$

The Euler's zeta function is an algebraic replica of Eratosthenes sieve

$$\zeta(s) \equiv \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n^{-s} \quad \text{for } \Re s = \sigma > 0. \tag{2}$$

Multiplying $\zeta(s)$ by the factor $1/2^s$, and subtracting from $\zeta(s)$, removes all the even terms $\propto 1/(2n)^s$ (e.g., $1/2^s + 1/4^s + 1/6^s + 1/8^s + \cdots$)

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots\right), \tag{3}$$

results in

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Likewise

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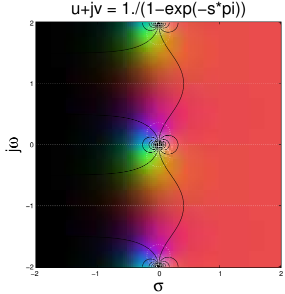
Euler's sieve gives Euler's product formula of $\zeta(s)$

$$\zeta(s) = \prod_{\pi_k \in \mathbb{P}} \frac{1}{1 - \pi_k^{-s}} = \prod_{\pi_k \in \mathbb{P}} \zeta_k(s), \tag{5}$$

where π_k represents the k^{th} prime. The above defines each prime factor

$$\zeta_k(s) = \frac{1}{1 - \pi_k^{-s}} = \frac{1}{1 - e^{-s \ln \pi_k}} \tag{6}$$



Plot of $w(s) = \frac{1}{1-e^{-s \ln \pi_1}}$ ($\pi_1 = 2$), factor $\zeta_1(s)$ (Eq. 5), which has poles where $e^{s_n \ln 2} = 1$, namely where $\omega_n \ln 2 = n2\pi$, as demonstrated by the domain–color map. $s = \sigma + \omega \jmath$ is the Laplace frequency.

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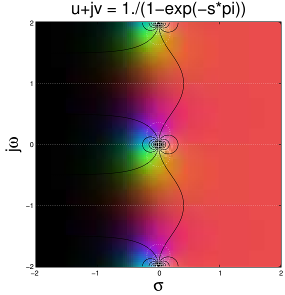

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Stream 3a: WEEK 8-12, Lects 23-34

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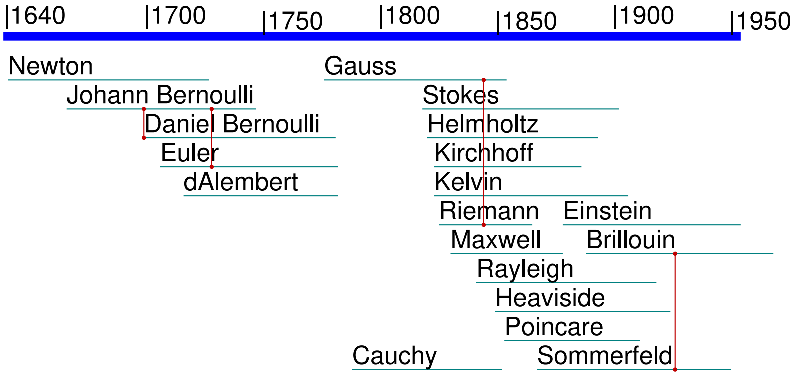
Time-line Newton-Einstein 1640-1950



- Notes:
  - Gaussian gap: Euler ⇒ Helmholtz
  - Connection between Gauss & Riemann
  - Heritage: Stokes & Helmholtz ⇒ Sommerfeld & Einstein

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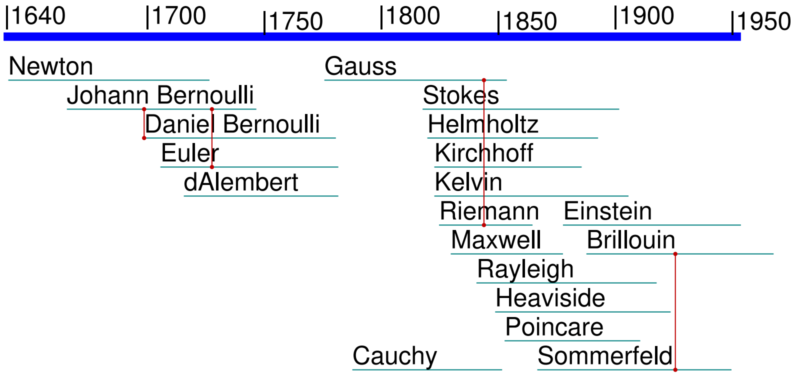

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Complex Analytic functions and Taylor series

An *analytic function* is one that may be expanded in a complex power series. Replace $x \in \mathbb{R}$ with $z = x + iy \in \mathbb{C}$

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- However the more general Taylor series has a problem: $z, F(z) \in \mathbb{C}$

$$F(z) = \sum_n \underbrace{\frac{1}{n!}\frac{d^n}{dz^n} f(z)}_{c_n} \Bigg|_{x=0} z^n$$

What does it mean to differentiate wrt $z \in \mathbb{C}$?

$$\frac{d}{dz}F(z) = \frac{d}{d(x+y\jmath)} F(x+y\jmath) \quad ???$$

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Complex analytic functions to solve difference equations

- Define *Laplace frequency* $s = \sigma + \omega \jmath$
- If

$$e^{st} = \sum_{n=1}^{\infty} \frac{1}{n!}(st)^n$$

- then

$$\frac{d}{dt} e^{st} = s e^{st}$$

  - e^{st} is an *eigenvector* of $\frac{d}{dt}$

d'Alembert: Creative, prolific & respected

Mapping the multi-valued square root of $w = \pm\sqrt{x+iy}$

- This provides a deep (essential) analytic insight.

15.3 Branch Points 303

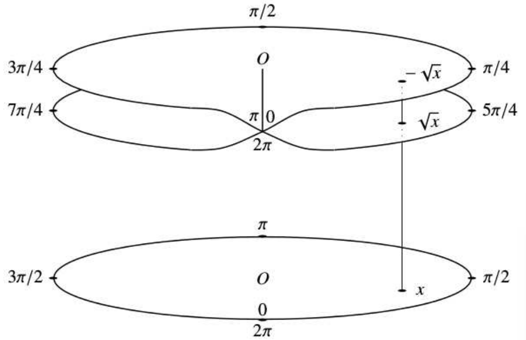

Figure 15.6: Branch point for the square root

- The Riemann Surface of the cubic $y^2 = x(x-a)(x-b)$ has Genis 1 (torus) (p. 307). *Elliptic functions* naturally follow.

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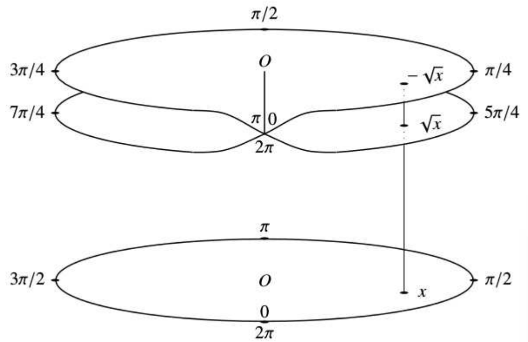

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Mapping complex analytic function

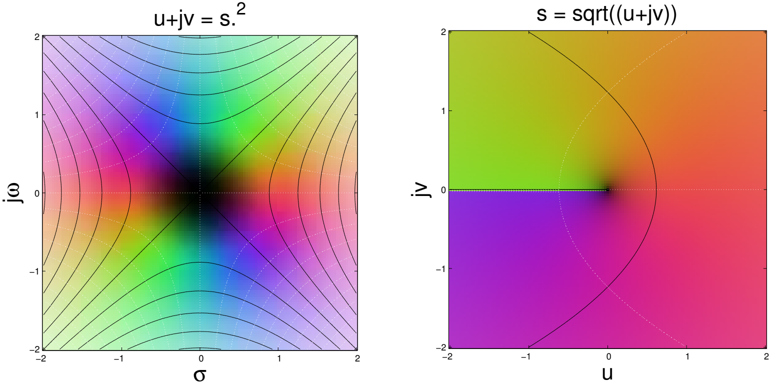


Figure: *Here the Cartesian coordinate map between $s = \sigma + \omega\jmath$ and $w = u + v\jmath$. LEFT: This shows the mapping $w(s) = s^2$. RIGHT: This shows the lower branch of the inverse $s(w) = \sqrt{w}$.*

Mapping complex analytic function

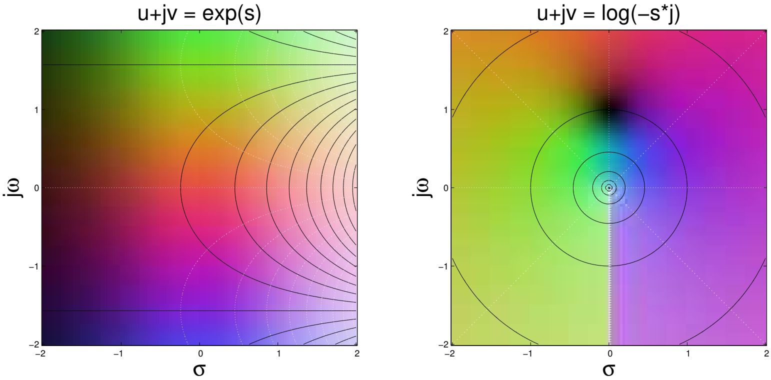
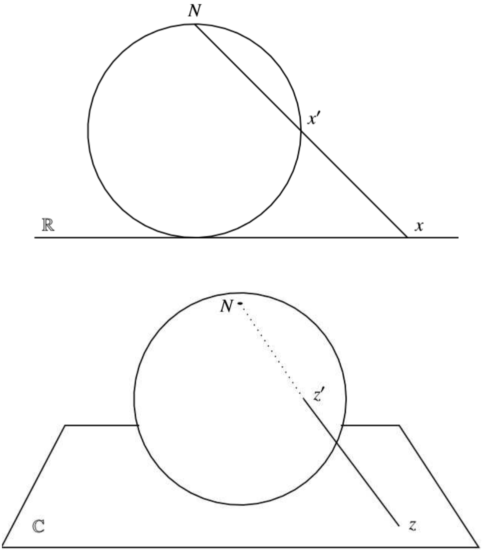


Figure: *Plots in the complex* $z = x + y\jmath$*: Left:* $e^{-s\jmath}$ *Right:* $\log(-s\jmath)$*, the inverse of the periodic* $e^{-s\jmath} = \cosh(-s\jmath) + \sinh(-s\jmath)$*, thus it has a branch cut, and a zero at* $s = \pi\jmath$ *(i.e.,* $\log(\pi\jmath = 0)$*).*

Riemann projection closes point $|z| \to \infty$ (i.e., $z' \to N$)



History of Acoustics, Music, Speech

- BC Pythagoras; Aristotle
- 16^{th} Mersenne, Marin 1588-1647; *Harmonie Universelle 1636*, *Father of acoustics*; Galilei, Galileo, 1564-1642; *Frequency Equivalence 1638*
- 17^{th} Newton, Hooke, Boyle
- 18^{th} Euler; d'Alembert; Gauss
- 19^{th} Fourier; Helmholtz; Kirchhoff; AG Bell; Lord Rayleigh

Stream 3b: WEEK 12-14, Lects 35-42

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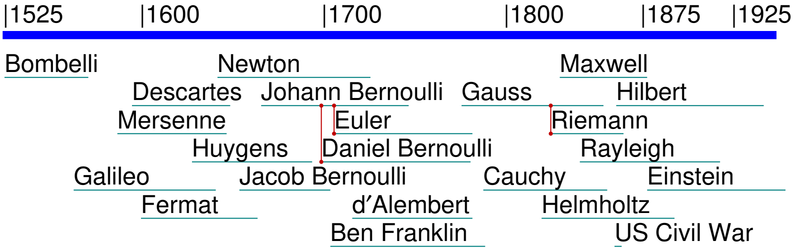

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  - $\Rightarrow$ Galileo, Descartes, Newton, Fermat, Bernoulli, Gauss, ...
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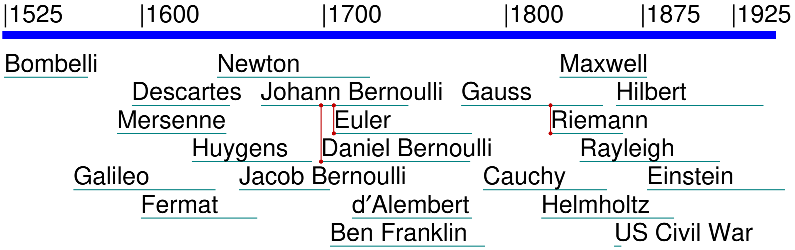

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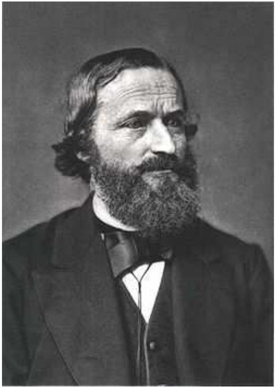
Acoustics, Vector calculus and circuit theory

- Helmholtz Theorem: Vector field $F = -\nabla\Phi + \nabla \times A$
- Kirchhoff's Laws of circuit theory (similar to Newton's Laws)

von Helmholtz

Gustav Kirchhoff



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